A Study on Approximate Reasoning Mechanisms via Fuzzy Relation Equations

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ABSTRACT

Some aspects of mechanisms of approximate reasoning and knowledge acquisition for rule-based expert systems are studied in the framework of fuzzy relation equations used for the implementation of inference procedures.

KEYWORDS: ***expert systems, fuzzy relation equation, production rules***

1. INTRODUCTION

The problem of processing linguistic and vague information in expert systems has been considered in diverse ways. It has been claimed that, in general, the uncertainty of input information and/or data forming a knowledge base cannot be neglected; nevertheless it is not obvious which tool should be applied. Some designers (e.g., Duda et al. [1]) prefer probabilistic schemes, especially a Bayesian scheme of reasoning. On the other hand, some well-known studies on human ways of data processing and human interference procedures (Coombs et al. [2]) lead to questions concerning the relevancy of the probabilistic approach proposed in expert systems. Therefore it is not surprising that one tries to follow some idea of subjective probability, upper and lower probabilities, the Dempster–Shafer theory of evidence, or some other mechanism that has been introduced to cope with the expert way of thinking in a "natural" manner (cf. certainty factors and their combination; Shortliffe [3]).

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Received October 1, 1987; accepted February 15, 1991.

International Journal of Approximate Reasoning 1992; 6:33–44

655 Avenue of the Americas, New York, NY 10010 0888-613X/92/$3.50

Fuzzy sets, especially their possibilistic interpretation, were proposed to handle uncertain information (Zadeh [4]). Despite many steps in this direction, in this paper we shall try to be as constructive as possible—to propose some mechanisms for expert systems and to provide methods potentially suitable for testing these mechanisms.

Unfortunately, which is astonishing nowadays using fuzzy set theory, there is an irritating trend to present generalized ideas without taking the care to validate them or even to show a reasonable method that enables others to check them. We study here expert systems based on production rules (Shortliffe [3]; Davis et al. [5]), which form an important class of application-oriented systems of knowledge engineering.

Our aim is to show how the mechanisms for combining pieces of evidence into the knowledge base and inferring are closely related.

The main background is the theory of fuzzy relation equations (Di Nola and coworkers [6, 7], Gottwald and Pedrycz [8], Miyakoshi and Shimbo [9], Pedrycz [10, 11]; Sanchez [12]). Since many results exist in this field, we refer the reader to the existing literature.

2. PRODUCTION RULES

Usually, by the production rules of an expert system we mean the triple

$$\langle A_i \rightarrow B_i,\ i = 1, 2, \ldots, n;\ \mathscr{C}, \mathscr{I} \rangle \tag{1}$$

where $A_i \rightarrow B_i$ denotes the ith production rule, $i = 1, 2, \ldots, n$, A_i being an antecedent formed by ANDing some subparts (subconditions),

$$A_i = A_{i1} \& A_{i2} \& \cdots \& A_{im} \tag{2}$$

with $A_{i1}, A_{i2}, \ldots, A_{im}$ specified as fuzzy sets,

$$A_{i1}\colon \mathscr{A}_1 \rightarrow [0, 1], \qquad A_{i2}\colon \mathscr{A}_2 \rightarrow [0, 1], \qquad \cdots \qquad , A_{im}\colon \mathscr{A}_m \rightarrow [0, 1]$$

the $\mathscr{A}_j$, $j = 1, 2, \ldots, m$, being suitable finite referential sets. In (1), B_i stands for the consequence, and $\mathscr{C}$ and $\mathscr{I}$ express symbolically combining and inference procedures, respectively.

Both A_i and B_i contain some fuzzy concepts, and therefore it is natural to represent them by fuzzy sets. In the discussion that follows, we consider A_i given by (2) and equal to the fuzzy relation

$$A_i = A_{i1} \times A_{i2} \times \cdots \times A_{im}$$

Thus,

$$A_i\colon \mathscr{A}_1 \times \mathscr{A}_2 \times \cdots \times \mathscr{A}_m \rightarrow [0, 1] \tag{3}$$

or, concisely,

$$A_i: \mathscr{A} \to [0, 1]$$

where

$$\mathscr{A} = \bigtimes_{j=1}^{m} \mathscr{A}_j$$

In terms of membership functions, (3) is read as

$$A_i(\mathbf{a}) = A_i(a_1, a_2, \ldots, a_m) = \bigwedge_{j=1}^{m} A_{ij}(a_j)$$

where $\mathbf{a} = (a_1, a_2, \ldots, a_m) \in \mathscr{A}$, $a_j \in \mathscr{A}_j$, $j = 1, 2, \ldots, m$, and $\wedge$ stands for the classical minimum operator of the unit interval [0, 1].

A fuzzy consequence is denoted by B and is defined in an appropriate space $\mathscr{B}$, $B: \mathscr{B} \to [0, 1]$. The ith production rule can be read as the if-then statement

$$\text{If } A_i \text{ then } B_i, \; i = 1, 2, \ldots, n$$

The combining procedure concerns a process of joining all the pieces of evidence $A_1, B_1, A_2, B_2, \ldots, A_n, B_n$ into the fuzzy relation R defined in the Cartesian product $\mathscr{A} \times \mathscr{B}$,

$$R: \mathscr{A} \times \mathscr{B} \to [0, 1]$$

Then the inference procedure deduces a consequence B for any antecedent A, $A: \mathscr{A} \to [0, 1]$ specified previously. More schematically, we put down

$$\begin{array}{ll} A & \\ R & (A_i \to B_i, \; i = 1, 2, \ldots, n) \\ \hline B & \end{array} \tag{4}$$

Bearing in mind that production rules are joint by the OR connective, the fuzzy relation R is treated as a union of partial results. More precisely,

$$R = \bigcup_{i=1}^{n} R_i, \qquad R(\mathbf{a}, b) = \bigvee_{i=1}^{n} R_i(\mathbf{a}, b), \qquad \mathbf{a} \in \mathscr{A}; \, b \in \mathscr{B} \tag{5}$$

where $\vee$ is the maximum operator of [0, 1] and $R_i = (A_i \Rightarrow B_i)$, which is read as

$$R_i(\mathbf{a}, b) = (A_i(\mathbf{a}) \Rightarrow B_i(b)), \qquad \mathbf{a} \in \mathscr{A}; \, b \in \mathscr{B}$$

where $\Rightarrow$ is any implication operator; usually the Lukasiewicz one is preferred. Nevertheless one can also view R_i as the Cartesian products of A_i's and B_i's (Mamdani and Assilian [13]), that is, $R_i = A_i \times B_i$. For the inference engine

(4), one takes the max-min composition of R and A,

$$B = A \circ R$$

which is read as

$$B(b) = \bigvee_{\mathbf{a}} \left[A(\mathbf{a}) \wedge R(\mathbf{a}, b) \right] \tag{6}$$

for any $b \in \mathscr{B}$. In the language of logics and a well-known interpretation of quantifiers $\exists$ and $\forall$, (6) is read as

$$\forall b \in \mathscr{B}, \exists \mathbf{a} \in \mathscr{A} : \left(A(\mathbf{a}) \wedge r_b(\mathbf{a}) \right) = B(b)$$

where

$$r_b(\mathbf{a}) = R(\mathbf{a}, b), \qquad r_b\colon \mathscr{A} \to [0, 1]$$

This means that $B(b)$ is equal to the degree to which A is equal to r_b (equal to a height of intersection of the fuzzy relations A and r_b).

Now we recall some fundamental results of the theory of fuzzy relation equations.

Defining the set

$$\mathscr{R}_i = \{R\colon B_i = A_i \circ R\}, \qquad i = 1, 2, \ldots, n$$

we have the following proposition.

Proposition 1 (Sanchez [12]) *If* $\mathscr{R}_i \neq \varnothing$, *then the fuzzy relation* $\hat{R}_i = A_i \textcircled{\alpha} B_i$ *with membership function given by*

$$\hat{R}_i(\mathbf{a}, b) = A_i(\mathbf{a}) \,\alpha\, B_i(b) = \begin{cases} 1 & \text{if } A_i(\mathbf{a}) \leq B_i(b) \\ B_i(b) & \text{otherwise} \end{cases}$$

is the greatest element of $\mathscr{R}_i$; *that is,* $\hat{R}_i(\mathbf{a}, b) \geq R(\mathbf{a}, b)$ *for any* $R \in \mathscr{R}_i$ *and for any* $\mathbf{a} \in \mathscr{A}$, $b \in \mathscr{B}$.

Proposition 2 (Gottwald [14]) *If the set*

$$\mathscr{R} = \bigcap_{i=1}^{n} \mathscr{R}_i \neq \varnothing \tag{7}$$

then the fuzzy relation

$$\hat{R} = \bigcap_{i=1}^{n} \hat{R}_i, \qquad \hat{R}(\mathbf{a}, b) = \bigwedge_{i=1}^{n} \hat{R}_i(\mathbf{a}, b), \qquad \mathbf{a} \in \mathscr{A};\, b \in \mathscr{B} \tag{8}$$

belongs to the set $\mathscr{R}$ *and is the greatest element of* $\mathscr{R}$.

It is evident that Proposition 2 gives a straightforward way to calculate the fuzzy relation of the knowledge base. Note that in the field of logic, the operator α, which appeared in Proposition 1, is the Gödel implication $\underset{G}{\Rightarrow}$,

$$x \alpha y = \left(x \underset{G}{\Rightarrow} y \right) = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } x > y \end{cases}$$

where x, $y \in [0, 1]$ are truth values of two propositions evaluated in many-valued logics.

The essence of the testing stage of the combination and inference mechanisms introduced earlier can be concisely summarized as follows. Take the antecedent as equal to A_i, $i = 1, 2, \ldots, n$, and check whether the resulting consequence B is equal to B_i. The closer B is to B_i, the better the combination and inference mechanisms are suited to the knowledge considered.

In case the combination of the pieces of evidence is realized by (5) and the inference is performed according to (6), the following remark is of practical interest (Gottwald and Pedrycz [15]). If A_i are normal fuzzy relations, then $B = A_i \circ R$ is a superset of B_i, that is, $B \supseteq B_i$. In fact, if $A_i(\mathbf{a}_0) = 1$ for some $\mathbf{a}_0$, then

$$\begin{aligned} B(b) = (A_i \circ R)(b) &= \bigvee_{\mathbf{a}} \left[A_i(\mathbf{a}) \wedge R(\mathbf{a}, b) \right] \\ &= \bigvee_{\mathbf{a}} \left\{ A_i(\mathbf{a}) \wedge \left[\bigvee_{j=1}^{n} \left(A_j(\mathbf{a}) \;\alpha\; B_j(b) \right) \right] \right\} \\ &\geq A_i(\mathbf{a}_0) \wedge \left[A_i(\mathbf{a}_0) \;\alpha\; B_i(b) \right] = 1 \wedge \left[1 \;\alpha\; B_i(b) \right] = B_i(b) \end{aligned}$$

Proposition 3 *If the knowledge base is constructed according to* (8) *and the inference mechanism is implemented by* (6), *then we have*

$$B \subseteq \bigcap_{i=1}^{n} \left[\pi(A_i / A) \;\alpha\; B_i \right]$$

where $\pi(A_i, A)$ *is the fuzzy set* (*constant with respect to* b) *with membership function given by*

$$\pi(A_i / A)(b) = \bigwedge_{\mathbf{a}} \left[A_i(\mathbf{a}) \wedge A(\mathbf{a}) \right]$$

for any $b \in \mathscr{B}$.

Proof We first notice that

$$a \wedge (b \;\alpha\; c) \leq (a \wedge b) \;\alpha\; c, \qquad \max_k \left(a_k \;\alpha\; b \right) = \left(\min_k a_k \right) \alpha\; b \quad (9)$$

where a, b, c, $a_k \in [0, 1]$. Using (9), we obtain, by performing the max-min composition for any $b \in \mathscr{B}$.

$$
\begin{aligned}
B(b) &= \bigvee_{\mathbf{a}} [A(\mathbf{a}) \wedge \hat{R}(\mathbf{a}, b)] = \bigvee_{\mathbf{a}} \left\{ A(\mathbf{a}) \wedge \left[\bigwedge_{i=1}^{n} (A_i(\mathbf{a}) \,\alpha\, B_i(b)) \right] \right\} \\
&= \bigwedge_{i=1}^{n} \left\{ \bigvee_{\mathbf{a}} [A(\mathbf{a}) \wedge (A_i(\mathbf{a}) \,\alpha\, B_i(b))] \right\} \\
&\le \bigwedge_{i=1}^{n} \left\{ \bigvee_{\mathbf{a}} [(A(\mathbf{a}) \wedge A_i(\mathbf{a})) \,\alpha\, B_i(b)] \right\} \\
&= \bigwedge_{i=1}^{n} \left\{ \left[\bigwedge_{\mathbf{a}} (A(a) \wedge A_i(\mathbf{a})) \right] \alpha\, B_i(b) \right\} \\
&= \bigwedge_{i=1}^{n} [\pi(A_i/A)(b) \,\alpha\, B_i(b)]
\end{aligned}
$$

3. FURTHER FORMS OF FUZZY RELATION EQUATIONS

Two forms of fuzzy relation equations can also be studied. The first one, studied by Miyakoshi and Shimbo [9], takes the form

$$B = A \, ⓐ \, R \tag{10}$$

which is read as

$$B(b) = \bigwedge_{\mathbf{a}} [A(\mathbf{a}) \,\alpha\, R(\mathbf{a}, b)]$$

for any $b \in \mathscr{B}$. Defining the set

$$\mathscr{R}_i' = \left\{ R \colon B_i = A_i \, ⓐ \, R \right\}$$

we have the following proposition.

PROPOSITION 4 (Miyakoshi and Shimbo [9]) *If* $\mathscr{R}_i' \neq \emptyset$, *then the fuzzy relation* $\check{R}_i = A_i \times B_i$ *with membership function given by*

$$\check{R}_i(\mathbf{a}, b) = A_i(\mathbf{a}) \wedge B_i(b)$$

is the smallest element of $\mathscr{R}_i'$; *that is,* $\check{R}_i(\mathbf{a}, b) \le R(\mathbf{a}, b)$ *for* $R \in \mathscr{R}_i'$ *and for any* $\mathbf{a} \in \mathscr{A}$, $b \in \mathscr{B}$.

Under a different approach, this result was also given by Di Nola et al. [7], where they established the following result.

PROPOSITION 5 *If the set*

$$\mathscr{R}' = \bigcap_{i=1}^{n} \mathscr{R}_i' \neq \varnothing \tag{11}$$

then the fuzzy relation

$$\check{R} = \bigcup_{i=1}^{n} \check{R}_i, \qquad \check{R}(\mathbf{a}, b) = \bigvee_{i=1}^{n} \check{R}_i(\mathbf{a}, b), \qquad \mathbf{a} \in \mathscr{A};\ b \in \mathscr{B}$$

belongs to the set $\mathscr{R}'$ *and is the smallest element of* $\mathscr{R}'$.

The second form of fuzzy relation equation, also studied in Ref. 9, is given by

$$B = R \circledA A \tag{12}$$

which is read as

$$B(b) = \bigwedge_{\mathbf{a}} \left[R(\mathbf{a}, b)\ \alpha\ A(\mathbf{a}) \right]$$

for any $b \in \mathscr{B}$. Defining the set

$$\mathscr{R}_i'' = \left\{ R\colon B_i = R \circledA A_i \right\}$$

we have the following.

PROPOSITION 6 (Miyakoshi and Shimbo [9]) *If* $R_i'' \neq \varnothing$, *then the fuzzy relation* $\tilde{R}_i = B_i \circledA A_i$ *with membership function given by*

$$\tilde{R}_i(a, b) = A_i(\mathbf{a}) \circledA B_i(b)$$

is the greatest element of $\mathscr{R}_i''$; *that is,* $R(\mathbf{a}, b) \leq \tilde{R}_i(\mathbf{a}, b)$ *for any* $R \in \mathscr{R}_i''$ *and for any* $\mathbf{a} \in \mathscr{A}$, $b \in \mathscr{B}$.

We now show the following result.

PROPOSITION 7 *If the set*

$$\mathscr{R}'' = \bigcap_{i=1}^{n} R_i'' \neq \varnothing \tag{13}$$

then the fuzzy relation

$$\tilde{R} = \bigcap_{i=1}^{n} \tilde{R}_i, \qquad \tilde{R}(\mathbf{a}, b) = \bigwedge_{i=1}^{n} \tilde{R}_i(\mathbf{a}, b), \qquad \mathbf{a} \in \mathscr{A};\ b \in \mathscr{B}$$

belongs to the set $\mathscr{R}''$ *and is the greatest element of* $\mathscr{R}''$.

Proof We first observe that the following property holds:

$$\text{If } a \le b, \text{ then } a \,\alpha\, c \ge b \,\alpha\, c \text{ for } a, b, c \in [0, 1]$$

Let R be an element of $\mathscr{R}''$. Hence $R \in \mathscr{R}_i''$ for any $i = 1, 2, \ldots, n$, and then $R \subseteq R_i$ for any $i = 1, 2, \ldots,$ by Proposition 6. This means that

$$R \subseteq \bigcap_{i=1}^{n} \tilde{R}_i = \tilde{R}$$

Thus, by the above property,

$$R(\mathbf{a}, b) \,\alpha\, A_i(\mathbf{a}) \ge \tilde{R}(\mathbf{a}, b) \,\alpha\, A_i(\mathbf{a}) \ge \tilde{R}_i(\mathbf{a}, b) \,\alpha\, A_i(\mathbf{a})$$

for any $i = 1, 2, \ldots, n$ and for any $\mathbf{a} \in \mathscr{A}$, $b \in \mathscr{B}$. This implies

$$\begin{aligned} B_i(b) &= \bigwedge_{\mathbf{a}} \left[R(\mathbf{a}, b) \,\alpha\, A_i(\mathbf{a}) \right] \ge \bigwedge_{\mathbf{a}} \left[\tilde{R}(\mathbf{a}, b) \,\alpha\, A_i(\mathbf{a}) \right] \\ &\ge \bigwedge_{\mathbf{a}} \left[\tilde{R}_i(\mathbf{a}, b) \,\alpha\, A_i(\mathbf{a}) \right] = B_i(b) \end{aligned}$$

for any $b \in \mathscr{B}$. That is,

$$B_i = R \circledast A_i \supseteq \tilde{R} \circledast A_i \supseteq \tilde{R}_i \circledast A_i = B_i$$

and then $\tilde{R} \circledast A_i = B_i$ for any $i = 1, 2, \ldots, n$. Thus $\tilde{R} \in \mathscr{R}''$. ∎

It is worthwhile to notice the logical interpretation of the forms of equations discussed in this section. Equation (10) is read as

$$B(b) = \forall \mathbf{a} \left[A(\mathbf{a}) \underset{G}{\Rightarrow} r_b(\mathbf{a}) \right]$$

Here the minimum is interpreted by the quantifier $\forall$, and interpreting $\underset{G}{\Rightarrow}$ as set inclusion we also have

$$B(b) = [A \subseteq r_b]$$

which, in other words, expresses $B(b)$ to the extent to which the fuzzy set A is contained in the fuzzy set r_b. $B(b)$ attains the value 1 if A is contained in at least one piece of evidence existing in the knowledge base expressed by the fuzzy relation R (inheritance property).

For Equation (12), we have a different situation:

$$B(b) = \forall \mathbf{a} \left[r_b(\mathbf{a}) \underset{G}{\Rightarrow} A(\mathbf{a}) \right]$$

that is,

$$B(b) = [r_b \subseteq A]$$

This means that $B(b)$ is equal to the degree of containment of the fuzzy set r_b in the piece of evidence A. To achieve the highest value of $B(b)$, r_b should inherit the same property that is met in A.

4. SOLVABILITY OF A SYSTEM OF FUZZY EQUATIONS

One of the crucial points that cannot be neglected is closely related to the problem of existence of solutions. Anyway, the assumption (7) either (11) or (13) is quite strong and difficult to satisfy. This fact was underscored at a quite early stage of development of fuzzy relation equations; see, for instance, Pedrycz [10]. Further studies led to an interesting interpretation of the results in terms of Φ-fuzzy sets (interval fuzzy sets) (Gottwald and Pedrycz [8]; Sambuc [16]).

We now suppose that the set of production rules are written as

$$B_i = A_i \circ R, \qquad i = 1, 2, \ldots, n \tag{14}$$

where R is a fuzzy relation combining all of them. The following result holds.

PROPOSITION 8 *If $\mathcal{R}_h \neq \emptyset$ for any $h = 1, 2, \ldots, n$, and A_i are pairwise disjoint—that is, $A_i(\mathbf{a}) \wedge A_h(\mathbf{a}) = 0$ for $i \neq h$ and for any $\mathbf{a} \in \mathcal{A}$, i, $h = 1, 2, \ldots, n$—then system* (14) *has solutions, that is, $\mathcal{R} \neq \emptyset$.*

Proof Let

$$\operatorname{supp} A_i = \{\mathbf{a} \in \mathcal{A} : A_i(\mathbf{a}) > 0\}$$

By virtue of the assumptions, we have

$$\operatorname{supp} A_i \cap \operatorname{supp} A_h = \emptyset$$

for any i, $h = 1, 2, \ldots, n$ such that $i \neq h$. Using the symbology of Proposition 2, we have

$$\hat{R}(\mathbf{a}, b) = \bigwedge_{i=1}^{n} \hat{R}_i(\mathbf{a}, b) = \hat{R}_h(\mathbf{a}, b) \wedge R'(\mathbf{a}, b)$$

for any $\mathbf{a} \in \mathcal{A}$, $b \in \mathcal{B}$, where

$$\hat{R}_h(\mathbf{a}, b) = A_h(\mathbf{a}) \,\alpha\, B_h(b), \quad R'(\mathbf{a},b) = \bigwedge_{i \neq h} \hat{R}_i(\mathbf{a}, b).$$

Since $A_i(\mathbf{a}) = 0$ for $i \neq h$ and for any $\mathbf{a} \in \operatorname{supp} A_h$, we deduce

$$R'(\mathbf{a}, b) = \bigwedge_{i \neq h} [A_i(\mathbf{a}) \,\alpha\, B_i(b)] = \bigwedge_{i \neq h} [0 \,\alpha\, B_i(b)] = 1$$

Further, we have that evidently

$$\bigvee_{\mathbf{a} \notin \operatorname{supp} A_h} [A_h(\mathbf{a}) \wedge \hat{R}_h(\mathbf{a}, b)] = 0$$

Since $\mathcal{R}_h \neq \emptyset$ for any $h = 1, 2, \ldots, n$, we obtain

$$\begin{aligned}
\bigvee_{\mathbf{a}} [A_h(\mathbf{a}) \wedge \hat{R}(\mathbf{a}, b)] &= \bigvee_{\mathbf{a}} [A_h(\mathbf{a}) \wedge \hat{R}_h(\mathbf{a}, b) \wedge R'(\mathbf{a}, b)] \\
&= \left\{ \bigvee_{\mathbf{a} \in \operatorname{supp} A_h} [A_h(\mathbf{a}) \wedge \hat{R}_h(\mathbf{a}, b) \wedge R'(\mathbf{a}, b)] \right\} \\
&\quad \vee \left\{ \bigvee_{\mathbf{a} \notin \operatorname{supp} A_h} [A_h(\mathbf{a}) \wedge \hat{R}_h(\mathbf{a}, b) \wedge R'(\mathbf{a}, b)] \right\} \\
&= \left\{ \bigvee_{\mathbf{a} \in \operatorname{supp} A_h} [A_h(\mathbf{a}) \wedge \hat{R}_h(\mathbf{a}, b)] \right\} \\
&\quad \vee \left\{ \bigvee_{\mathbf{a} \notin \operatorname{supp} A_h} [0 \wedge \hat{R}_h(\mathbf{a}, b) \wedge R'(\mathbf{a}, b)] \right\} \\
&= \left\{ \bigvee_{\mathbf{a} \in \operatorname{supp} A_h} [A_h(\mathbf{a}) \wedge \hat{R}_h(\mathbf{a}, b)] \right\} \\
&= \left\{ \bigvee_{\mathbf{a} \in \operatorname{supp} A_h} [A_h(\mathbf{a}) \wedge \hat{R}_h(\mathbf{a}, b)] \right\} \\
&\quad \vee \left\{ \bigvee_{\mathbf{a} \notin \operatorname{supp} A_h} [A_h(\mathbf{a}) \wedge \hat{R}_h(\mathbf{a}, b)] \right\} \\
&= \bigvee_{\mathbf{a}} [A_h(\mathbf{a}) \wedge \hat{R}_h(\mathbf{a}, b)] = B_h(b)
\end{aligned}$$

for any $h = 1, 2, \ldots, n$ and for any $b \in \mathcal{B}$. This means that the fuzzy relation $\hat{R}$ belongs to $\mathcal{R}_h$ for any $h = 1, 2, \ldots, n$, that is, $\mathcal{R} \neq \emptyset$ since $\hat{R} \in \mathcal{R}$. ∎

In practical cases, not all those A_i, B_i, $i = 1, 2, \ldots, n$, satisfy the Equation (14), for several reasons (Gottwald and Pedrycz [17]). Then one can modify the data A_i, B_i, imposing some threshold level, as is proved by Gottwald and Pedrycz [17].

5. CONCLUDING REMARKS

Selected topics in rule-based expert systems have been studied in a plausible setting of various types of fuzzy relation equations. The properties of the resulting inference engines are a consequence of the imposed structure of the type of equation.

The results of this paper are of a preliminary character. Further investigations are necessary and will be reported in future works.

ACKNOWLEDGMENT

We thank a referee for useful suggestions that have improved the presentation of this paper.

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